Generalized quadrature domains

Andrew Graven

Classical quadratur

Abelian quadrature domains

Weighted quadrature domains

Eutura worl

Generalized Quadrature Domains

with connections to Hele-Shaw flow

Andrew Graven
Caltech Department of Mathematics
(joint work with Nikolai Makarov)

Random Matrices and Related Topics in Jeju May 10, 2024

With support from



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Mean value property:

$$f\in L^1_a(\mathbb{D}_r(a)) \quad \Longrightarrow \quad rac{1}{\pi r^2} \int_{\mathbb{D}_r(a)} f dA = f(a).$$

¹bounded & simply connected

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The cardioid, $\Omega=\left\{z+rac{z^2}{2}:z\in\mathbb{D}\right\}$:

$$2 = \left\{ z + \frac{z}{2} : z \in \mathbb{D} \right\}$$

$$f \in L^1_a(\Omega) \implies \frac{1}{\pi} \int_{\Omega} f dA = \frac{3}{2} f(0) + \frac{1}{2} f'(0).$$

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Aharonov & Shapiro (1976): The cardioid is also unique.

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These are examples of *quadrature identities*.

¹bounded & simply connected

Eutura mari

Definition 1.1 (Quadrature domain)

We call a domain $\Omega \subset \widehat{\mathbb{C}}$ a quadrature domain if there exists $h \in \mathsf{Rat}(\Omega)$ s.t.²

$$rac{1}{\pi}\int_{\Omega}fdA=rac{1}{2\pi i}\oint_{\partial\Omega}f(w)h(w)dw$$

 $\forall f \in L^1_a(\Omega)$. This is denoted by $\Omega \in \mathsf{QD}(h)$. (we also assume $\infty \notin \partial \Omega$)

 $^{{}^{2}}$ Rat (Ω) = space of rational functions analytic in Ω^{c} . (all poles are in Ω)

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Residue theorem (Ω bounded) o quadrature domain \iff quadrature identity:

$$\frac{1}{2\pi i} \oint_{\partial \Omega} f(w)h(w)dw = \sum_{\text{poles of } h, \{p_k\}} \operatorname{Res}_{w=p_k}(f(w)h(w)) = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k).$$

Unbounded case is similar.

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 $\frac{1}{\pi}\int_{\Omega}fdA=\frac{1}{2\pi i}\oint_{\partial\Omega}f(w)h(w)dw$

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Residue theorem (Ω bounded) \rightarrow quadrature domain \iff quadrature identity:

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Unbounded case is similar. $rac{1}{\pi}\int_{\Omega}\mathit{fd}A=\mu(f),\quad ext{where } \mu=\overline{\partial}\mathit{h}=\sum_{\iota,\,:}\mathit{c}_{k,j}(-1)^{n_j+1}\delta_{p_k}^{(n_j)}$ Can also write

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Unbounded QD Example: The deltoid

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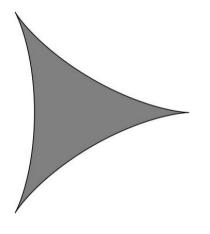
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The complement of the deltoid is an unbounded quadrature domain,

$$\Omega = \left\{ z + \frac{1}{2z^2} : |z| > 1 \right\} \in \mathsf{QD}\left(\frac{w^2}{2}\right) :$$

$$\frac{1}{\pi} \int_{\Omega} f dA = \frac{1}{2\pi i} \oint_{\partial \Omega} f(w) \frac{w^2}{2} dw$$



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$$\frac{1}{\pi} \int_{\Omega} f dA = \frac{1}{2\pi i} \oint_{\partial \Omega} f(w) \frac{w^2}{2} dw$$
$$= \frac{1}{2} f_3$$

$$f(w) = f_1 w^{-1} + f_2 w^{-2} + f_3 w^{-3} + \cdots$$

$$(f \in L_2^1 \implies f_1 = f_2 = 0)$$

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Definition 1.2 (Cauchy transform)

For a Borel set $\Omega \subset \mathbb{C}$, we denote the *Cauchy transform* of Ω by $C^{\Omega} : \mathbb{C} \to \mathbb{C}$,

$$C^{\Omega}(w) = \frac{1}{\pi} \int_{\Omega} \frac{dA(\xi)}{w - \xi}$$

 $\overline{C^{\Omega}}$ corresponds to the electric field due to a uniform charge distribution on Ω .

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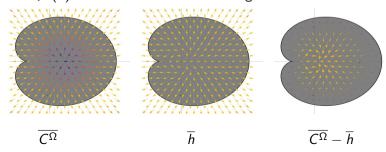
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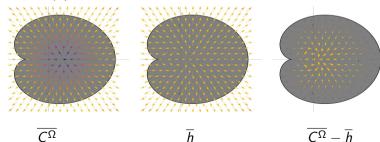
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 $\mu = \overline{\partial} h$ corresponds to point charge distribution.

Schwarz function

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Remark: $\Omega \subset \widehat{\mathbb{C}}$ is a QD iff it admits a *Schwarz function* $S : \Omega \to \widehat{\mathbb{C}}$.

$$C^{\Omega^c}(w) = \lim_{r \to \infty} \frac{1}{\pi} \int_{\Omega^c \cap \mathbb{D}_r} \frac{dA(\xi)}{w - \xi}$$

³ ≐ denotes equality on the boundary.

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Remark: $\Omega \subset \widehat{\mathbb{C}}$ is a QD iff it admits a *Schwarz function* $S : \Omega \to \widehat{\mathbb{C}}$.

A S-function is a continuous map

$$\mathcal{S}:\mathsf{Cl}(\Omega) o\widehat{\mathbb{C}}$$

such that $S \in \mathcal{M}(\Omega)$ and 3

$$S(w) \dot{=} \overline{w}$$

$$C^{\Omega^{c}}(w) = \lim_{r \to \infty} \frac{1}{\pi} \int_{\Omega^{c} \cap \mathbb{D}_{r}} \frac{dA(\xi)}{w - \xi}$$

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such that $S \in \mathcal{M}(\Omega)$ and³

Also. $S(w) = h(w) + C^{\Omega^c}(w), \quad w \in \Omega$

(where C^{Ω^c} is understood in terms of its Cauchy principal value)⁴

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boundary.
$$C^{\Omega^c}(w)$$

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 $S(w) = \overline{w}$

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Let $\Omega\subset\mathbb{C}$ be bounded and simply connected with Riemann map $\varphi:\mathbb{D}\to\Omega$,

$$\varphi(z)=f_0+f_1z+f_2z^2+\cdots$$

$${}^{5}\mathcal{C}_{A}(X) = \text{functions analytic in } X$$
, continuous up to ∂X , and $= 0$ at ∞ .

quadrature domains $(\psi = \varphi^{-1})$

Future work

Let $\Omega\subset\mathbb{C}$ be bounded and simply connected with Riemann map $\varphi:\mathbb{D}\to\Omega$,

$$\varphi(z)=f_0+f_1z+f_2z^2+\cdots$$

The associated interior Faber transform Φ_{φ} is a linear iso $\mathcal{C}_{A}(\mathbb{D}^{c}) o \mathcal{C}_{A}(\Omega^{c})$,⁵

$$\Phi_{\varphi}(f)(w) = \frac{1}{2\pi i} \oint_{\partial \mathbb{D}} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz = \frac{1}{2\pi i} \oint_{\partial \Omega} \frac{f \circ \psi(\xi)}{\xi - w} d\xi$$

$${}^{5}\mathcal{C}_{A}(X)=$$
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The definition of the exterior Faber transform is analogous.

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$$arphi(z)$$
 =

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 $\Phi_{\varphi}(f)(w) = \frac{1}{2\pi i} \oint_{\Omega^{m}} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz = \frac{1}{2\pi i} \oint_{\partial \Omega} \frac{f \circ \psi(\xi)}{\xi - w} d\xi$

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 $(\psi = \varphi^{-1})$ The definition of the exterior Faber transform is analogous.

$$\Phi_{\varphi}\left(\frac{1}{z-z_0}\right)(w) = \frac{\varphi'(z_0)}{w-\varphi(z_0)}, \qquad F_n = \Phi_{\varphi}\left(z^n\right) \text{ (nth Faber polynomial)}$$

$$\Phi_{\varphi}\left(\frac{1}{z}\right)$$

$$W-\varphi($$

$${}^{5}C_{A}(X) = \text{functions analytic in } X$$
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If $\Omega \in QD(h)$ is s.c, with Riemann map φ , then φ is rational and⁶

$$h=\Phi_{arphi}\left(arphi^{\#}
ight)$$

Chang & Makarov (2013)

$$\overline{{}^6arphi^\#(z)}:=\overline{arphi(\overline{z^{-1}})}$$

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Example:
$$\Omega \in \mathsf{QD}\left(\sum_{j=1}^n \frac{c_j}{(w-w_0)^j}\right)$$
 is bounded (wlog assume $\varphi(0)=w_0$),

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 \longrightarrow finite-dimensional system of algebraic equations relating φ and h.

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$$\implies \varphi(z) = w_0 + \sum_{i=1}^n \frac{\alpha_i}{(z^{-1} - \overline{\psi(w_0)})^j} = w_0 + \sum_{i=1}^n \alpha_i z^i$$

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 $^6\varphi^\#(z) := \varphi(\overline{z^{-1}})$

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domains

$$h = \Phi_{\varphi} \left(\varphi^{\#} \right)$$
 Chang & Makarov (2013)

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$$\implies \varphi(z) = w_0 + \sum_{i=1}^n \frac{\alpha_j}{(z^{-1} - \overline{\psi(w_0)})^j} = w_0 + \sum_{i=1}^n \alpha_i z^i$$

$$\implies \sum_{j=1}^{n} \frac{c_j}{(w-w_0)^j} = \Phi_{\varphi}\left(\overline{w_0} + \sum_{j=1}^{n} \overline{\alpha_j} z^{-j}\right)(w) = \sum_{j=1}^{n} \frac{q_j(\alpha, \overline{\alpha})}{(w-w_0)^j}$$

Future wor

Definition 2.1 (Abelian quadrature domain)

We call a bounded domain $\Omega \subset \mathbb{C}$ an Abelian quadrature domain if there exists $h \in \widehat{\mathsf{Rat}}(\Omega)$, h = r + L for some $r \in \mathsf{Rat}(\Omega)$ and $e^L \in \mathsf{Rat}(\Omega)$ such that

$$rac{1}{\pi}\int_{\Omega}fdA=rac{1}{2\pi i}\oint_{\partial\Omega}f(w)h(w)dw.$$

 $\forall f \in L^1_a(\Omega)$. This is denoted by $\Omega \in \widetilde{\mathsf{QD}}(h)$.

 $^{^{7}}h \in \widetilde{\mathsf{Rat}}(\Omega) \iff h \text{ is analytic in } \Omega^{c} \text{ with } h' \in \mathsf{Rat}(\Omega)$

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$$\pi J_{\Omega}$$
 $2\pi i J_{\partial\Omega}$

 $\forall f \in L^1_a(\Omega)$. This is denoted by $\Omega \in \overline{QD}(h)$.

$$\frac{1}{2\pi i} \oint_{\partial \Omega} f(w)h(w)dw = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k) - \sum_{i} \alpha_j \int_{a_j}^{b_j} f(w)dw$$

(where the a_i , b_i are the pairs of branch points of L)

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Definition 2.1 (Abelian quadrature domain)

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 $h \in \operatorname{Rat}(\Omega)$, h = r + L for some $r \in \operatorname{Rat}(\Omega)$ and $e^L \in \operatorname{Rat}(\Omega)$ such that

 $\frac{1}{\pi} \int_{\Omega} f dA = \frac{1}{2\pi i} \oint_{\partial \Omega} f(w) h(w) dw.$

 $\forall f \in L^1(\Omega)$. This is denoted by $\Omega \in \overline{\mathrm{QD}}(h)$.

Residue theorem:

 $\frac{1}{2\pi i} \oint_{\partial \Omega} f(w)h(w)dw = \sum_{k,i} c_{k,j} f^{(n_j)}(p_k) - \sum_{i} \alpha_j \int_{a_i}^{b_j} f(w)dw$

(where the a_i , b_i are the pairs of branch points of L)

Remark: The Faber transform formula also applies to Abelian QDs. ${}^{7}h \in \widetilde{\mathsf{Rat}}(\Omega) \iff h \text{ is analytic in } \Omega^{\mathsf{c}} \text{ with } h' \in \mathsf{Rat}(\Omega)$

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Abelian QD Example

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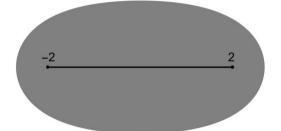
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The following ellipse-like region is an AQD:

$$\Omega = \left\{ w \in \mathbb{C} : \left| \mathsf{tanh}\left(\frac{w}{2}\right) \right|^2 < \mathsf{tanh}\left(1\right) \right\} \in \widetilde{\mathsf{QD}}\left(\mathsf{In}\left(\frac{w+2}{w-2}\right)\right)$$



$$\frac{1}{\pi} \int_{\Omega} f dA = \frac{1}{2\pi i} \oint_{\partial \Omega} f(w) \ln \left(\frac{w+2}{w-2} \right) dw = \int_{-2}^{2} f(w) dw$$

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Definition 3.1 (Weighted quadrature domain)

We call a domain $\Omega \subset \widehat{\mathbb{C}}$ a weighted quadrature domain wrt the weight $\rho: \Omega \to \mathbb{R}_{\geq 0}$ if $\exists h \in \mathsf{Rat}(\Omega)$ s.t

$$\frac{1}{\pi} \int_{\Omega} f \rho dA = \frac{1}{2\pi i} \oint_{\partial \Omega} f(w) h(w) dw$$

 $\forall f \in L^1_a(\Omega; \rho)$. This is denoted by $\Omega \in \mathsf{QD}_\rho(h)$.

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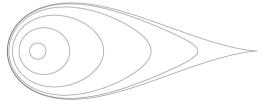
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Example: if $\varphi(z) = aze^{az^{-1}}$ (0 < $a \le 1$), then $\Omega = \varphi(\mathbb{D}^-) \in \mathsf{QD}_{|w|^{-2}}(1)$



 $(\Omega \text{ is unbounded component})$

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Recall that if $\Omega \in QD(h)$ (bounded, s.c.), then $h = \Phi_{\varphi}(\varphi^{\#}).^{8}$ This generalizes to certain classes of weighted QDs.

$$^8\varphi^\#(z)=\overline{\varphi(\overline{z}^{-1})}$$

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If $\Omega \in \mathrm{QD}_{\rho}(h)$ is bounded and s.c, with $\rho = |R'|^2 = \frac{\Delta |R|^2}{4}$, for $R \in \mathrm{\overline{Rat}}$ and $\infty, 0 \notin R'(\Omega)$, then

$$h(w) = \Phi_{\varphi}\left(R' \circ \varphi(z)(R \circ \varphi)^{\#}(z)\right)(w)$$

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And

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Remark 1: $\implies R \circ \varphi$ is a rational function.

$$^{8}\varphi^{\#}(z) = \overline{\varphi(\overline{z}^{-1})}$$

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Weighted Faber transform formula Recall that if $\Omega \in QD(h)$ (bounded, s.c.), then $h = \Phi_{\wp}(\varphi^{\#})$. This generalizes to

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certain classes of weighted QDs. If $\Omega \in QD_{\rho}(h)$ is bounded and s.c, with $\rho = |R'|^2 = \frac{\Delta |R|^2}{4}$, for $R \in \widetilde{Rat}$ and

 $h(w) = \Phi_{\varphi}\left(R'\circ \varphi(z)(R\circ \varphi)^{\#}(z)\right)(w)$

 $\left|R\circarphi(z)=R\circarphi(0)+\Phi_{arphi}^{-1}\left(rac{h}{R'}
ight)^{\#}(z)
ight|$

Remark 2: generalizes nicely to unbounded domains and those with $0, \infty \in R'(\Omega)$.

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⁹wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$ ¹⁰see [Dragnev, Legg & Saff (2022)] for a similar result

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$$R\circ arphi(z)=R\circ arphi(0)+\Phi_{arphi}^{-1}\left(rac{c}{(w-w_0)R'(w)}
ight)^{\#}(z)$$

 $^{^{9}}$ wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$ 10 see [Dragnev, Legg & Saff (2022)] for a similar result

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$$R \circ \varphi(z) = R \circ \varphi(0) + \Phi_{\varphi}^{-1} \left(\frac{c}{(w - w_0)R'(w)} \right)^{\#} (z)$$

$$= R(w_0) + \frac{c}{R'(w_0)} \Phi_{\varphi}^{-1} \left(\frac{1}{w - w_0} - \frac{R'(w) - R'(w_0)}{w - w_0} \frac{1}{R'(w)} \right)^{\#} (z)$$

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$$= R(w_0) + \frac{c}{R'(w_0)} \frac{\overline{\psi'(w_0)}}{z^{-1} - \overline{\psi(w_0)}} = R(w_0) + \alpha z.$$

⁹wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$ ¹⁰see [Dragnev, Legg & Saff (2022)] for a similar result

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If $\Omega\in\mathsf{QD}_{|R'|^2}\left(rac{c}{w-w_0}
ight)$ is bounded and s.c. with, c>0, $w_0\in\mathbb{C}$ "nice", then⁹

$$R \circ \varphi(z) = R \circ \varphi(0) + \Phi_{\varphi}^{-1} \left(\frac{c}{(w - w_0)R'(w)} \right)^{\#} (z)$$

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So Ω is a preimage of a disk under R.

⁹wlog taking $\varphi(0) = w_0, \ \varphi'(0) > 0$

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$$R\circ arphi($$

$$R\circ \varphi(z)$$

$$\circ \varphi(z)$$

can show that $\alpha = \sqrt{c}$.

⁹wlog taking $\varphi(0) = w_0, \varphi'(0) > 0$

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$$R \circ \varphi(z) = R \circ \varphi(0) + \Phi_{\varphi}^{-1} \left(\frac{c}{(w - w_0)R'(w)} \right)^{\#} (z)$$

$$\varphi(0) +$$

$$()+\Phi_{arphi}^{-1}\left(rac{\pi}{2}
ight)$$

$$= R(w_0) + \frac{c}{R'(w_0)} \Phi_{\varphi}^{-1} \left(\frac{1}{w - w_0} - \frac{R'(w) - R'(w_0)}{w - w_0} \frac{1}{R'(w)} \right)^{\#} (z)$$

$$\Phi_{\omega}^{-1}$$

$$lackbox{}_{arphi}^{-1}$$

$$^{\prime}arphi$$
 $_{-}$

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$$v(w_0)$$

 $c = \frac{1}{2\pi i} \oint_{\partial \Omega} 1 \cdot \frac{c}{w - w_0} dw = \frac{1}{\pi} \int_{\Omega} 1 \cdot |R'|^2 dA$

So
$$\Omega$$
 is a preimage of a disk under R . Using

$$v_0)$$

$$\overline{(w)}$$
 $\int_{-\infty}^{\infty} (2)$

with,
$$c>0$$
, $w_0\in\mathbb{C}$ "nice", th

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$$O \subset OD$$

If
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$$R \circ \varphi(z) = R \circ \varphi(0) + \Phi_{\varphi}^{-1} \left(\frac{c}{(w - w_0)R'(w)} \right)^{\#} (z)$$

$$(w - w_0)R'(w)$$

$$= R(w_0) + \frac{c}{R'(w_0)} \Phi_{\varphi}^{-1} \left(\frac{1}{w - w_0} - \frac{R'(w) - R'(w_0)}{w - w_0} \frac{1}{R'(w)} \right)^{\#} (z)$$

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$$2\pi I \ J_{\partial\Omega} \qquad W$$

can show that
$$\alpha = \sqrt{c}$$
. \longrightarrow Ω is a preimage of $\mathbb{D}_{\sqrt{c}}(R(w_0))$ under R^{10} . Ω is a preimage of $\mathbb{D}_{\sqrt{c}}(R(w_0))$ under R^{10} .

 9 wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$ ¹⁰see [Dragnev, Legg & Saff (2022)] for a similar result

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Future work

Classical Hele-Shaw flow:

ullet family of domains $\{\Omega_t\}_t$ in plane with smooth boundary

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Classical Hele-Shaw flow:

- ullet family of domains $\{\Omega_t\}_t$ in plane with smooth boundary
- boundary evolves with $v_n \propto \nabla G_\infty$ (v_n =normal velocity, G_∞ = Green function wrt ∞)

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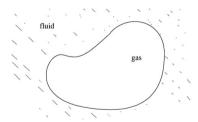
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Varchenko & Etingof (1992)

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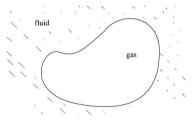
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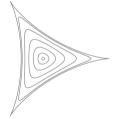
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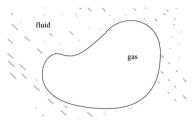
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Varchenko & Etingof (1992)

Deltoid

Remark: $\Omega_{t_0} \in QD(h) \implies \Omega_{t_0+\delta t} \in QD\left(h(w) + \frac{\delta t}{w-w_0}\right)$, where w_0 is the injection point.

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Finger of water penetrating oil¹¹

¹¹Saffman & Taylor (1958)

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Finger of water penetrating oil 11 Saffman-Taylor finger \longleftrightarrow Hele-Shaw cell in channel

¹¹Saffman & Taylor (1958)

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Finger of water penetrating oil¹¹

Saffman-Taylor finger \longleftrightarrow Hele-Shaw cell in channel

Recall:

 $\mathsf{quadrature}\ \mathsf{domain} \quad \longleftrightarrow \quad \mathsf{Hele}\text{-}\mathsf{Shaw}\ \mathsf{cell}$

¹¹Saffman & Taylor (1958)

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Finger of water penetrating oil¹¹

Saffman-Taylor finger \longleftrightarrow Hele-Shaw cell in channel

Recall:

quadrature domain \longleftrightarrow Hele-Shaw cell

So,

quadrature domain \longleftrightarrow Saffman-Taylor finger

¹¹Saffman & Taylor (1958)

Saffman-Taylor fingers

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Finger of water penetrating oil¹¹

Hele-Shaw cell in channel

Saffman-Taylor finger

Recall:

quadrature domain Hele-Shaw cell

quadrature domain

So,

Problem: ∞ in boundary

Saffman-Taylor finger

¹¹Saffman & Taylor (1958)

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Definition 3.2 (Weighted Abelian QD)

We call a bounded domain $\Omega \subset \mathbb{C}$ a weighted *Abelian quadrature domain* wrt the weights $\rho: \Omega \to \mathbb{R}_{\geq 0}$ and $\Lambda \in \operatorname{Rat}(\Omega^c)$ if $\exists h = r + \Lambda L$ for $r, e^L \in \operatorname{Rat}(\Omega)$, such that $\frac{1}{\pi} \int_{\Omega} f \rho dA = \frac{1}{2\pi i} \oint_{\partial \Omega} f(w) h(w) dw$

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 $\forall f \in L^1_a(\Omega; \rho)$. This is denoted by $\Omega \in \overline{\mathsf{QD}}_{\rho}(h)$.

Residue theorem:

$$\frac{1}{2\pi i} \oint_{\partial \Omega} f(w)h(w)dw = \sum_{k,j} c_{k,j} f^{(n_j)}(p_k) - \sum_j \alpha_j \int_{\mathsf{a}_j}^{\mathsf{b}_j} f(w)\Lambda(w)dw$$

(where the a_j , b_j are the pairs of branch points of L)

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(where the a_i , b_i are the pairs of branch points of L)

Unbounded case is similar.

Weighted Abelian QD Example

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If $0 \notin \Omega \in \mathsf{QD}_{|w|^{-2}}(h)$ is s.c, can still obtain a Faber transform formula

$$h(w) = rac{1}{w} \Phi_{arphi} \left(\ln \left(rac{arphi(z)}{arphi(0)}
ight)^{\#} \right) (w)$$

¹²pay no attention to the singularity on the boundary...

Weighted Abelian QD Example

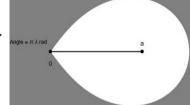
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If $0 \notin \Omega \in \widetilde{\mathsf{QD}}_{|w|-2}(h)$ is s.c, can still obtain a Faber transform formula

$$h(w) = rac{1}{w} \Phi_arphi \left(\ln \left(rac{arphi(z)}{arphi(0)}
ight)^\#
ight) (w) \hspace{1cm} a \mathbb{D}(1)^\lambda o Angle = \pi \lambda$$
 rad



Let
$$a\mathbb{D}(1)^{\lambda} \in \widetilde{\mathsf{QD}}_{|w|^{-2}}(h)$$
, 12 $a>0$, $\lambda \in (0,2]$

¹²pay no attention to the singularity on the boundary...

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ightarrow {}_{ ext{Angle} = \pi \lambda \, ext{rad}}$$



Let
$$a\mathbb{D}(1)^{\lambda} \in \widetilde{\mathsf{QD}}_{|w|^{-2}}(h)$$
, $a > 0$, $\lambda \in (0,2]$

$$\varphi(z)=a(z+1)^{\lambda}$$

¹²pay no attention to the singularity on the boundary...

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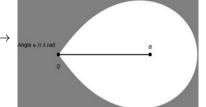
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ight)(w)} \quad a \mathbb{D}(1)^\lambda o Angle * \pi \lambda \, \mathrm{rad}}$$



Let $a\mathbb{D}(1)^{\lambda} \in \widetilde{\mathsf{QD}}_{|w|^{-2}}(h)$, 12 a > 0, $\lambda \in (0,2]$

$$arphi(z) = \mathsf{a}(z+1)^\lambda \quad \implies \quad \mathsf{h}(w) = rac{\lambda}{w} \Phi_{arphi} \left(\mathsf{ln} \left(rac{z+1}{z}
ight) \right) (w)$$

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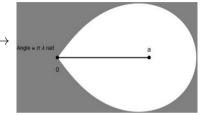
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$$arphi(z) = a(z+1)^{\lambda} \quad \implies \quad h(w) = rac{\lambda}{w} \Phi_{arphi} \left(\ln \left(rac{z+1}{z}
ight)
ight) (w) = rac{\lambda}{w} \ln \left(rac{w}{w-a}
ight)$$

¹²pay no attention to the singularity on the boundary...

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 $a \mathbb{D}(1)^{\lambda} o Angle * \pi \lambda \operatorname{rad}$

ngle ≈ π λ rad a

Let $a\mathbb{D}(1)^{\lambda} \in \widetilde{\mathsf{QD}}_{|w|^{-2}}(h)$, 12 a>0, $\lambda \in (0,2]$

So if
$$f \in L^1_2(a\mathbb{D}(1)^{\lambda}; |w|^{-2})$$
.

$$\varphi(z) = a(z+1)^{\lambda} \implies h(w) = \frac{\lambda}{w} \Phi_{\varphi} \left(\ln \left(\frac{z+1}{z} \right) \right) (w) = \frac{\lambda}{w} \ln \left(\frac{w}{w-a} \right)$$

$$\frac{1}{\pi} \int_{\mathbb{R}^{|\mathbb{D}(1)|\lambda}} \frac{f(w)}{|w|^2} dA(w) = \lambda \int_0^a \frac{f(w)}{w} dw$$

¹² pay no attention to the singularity on the boundary...

Weighted Abelian QD Example (cont.)

Generalized quadrature domains

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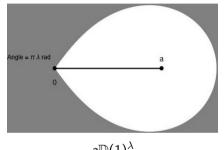
Classical quadratur domains

Abelian quadrature domains

Weighted quadrature domains

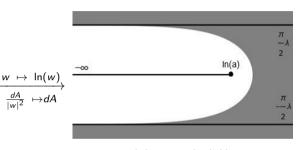
Future work

Changing variables $w \mapsto \ln(w)$:



 $a\mathbb{D}(1)^{\lambda}$

(Saffman-Taylor finger)



 $\mathsf{In}(a) + \lambda \, \mathsf{In}(\mathbb{D}(1))$

Weighted Abelian QD Example (cont.)

quadrature domains Andrew Graven

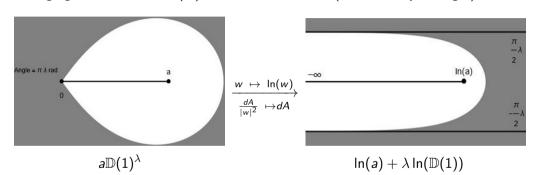
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quadrature

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Eutura war



(Saffman-Taylor finger)

Gives

$$\frac{1}{\pi} \int_{\ln(a) + \lambda \ln(\mathbb{D})} f dA = \lambda \int_{-\infty}^{\ln(a)} f(w) dw$$

For $f \in L^1_2(\ln(a) + \lambda \ln(\mathbb{D}(1)))$

Changing variables $w \mapsto \ln(w)$:

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Classical

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Weighted quadrature domains

Future work

• Generalize beyond simply connected domains?

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Future work

- Generalize beyond simply connected domains?
 - There are generalizations of Faber transform to multiply connected domains, but not as straightforward to work with

Generalized quadrature domains

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 - Classical case: Lee & Makarov (2016)

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Future work

Thank you!

One point unbounded quadrature domains

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Future work

Upcoming: Characterization of simply connected 1 pt UQDs,

$$\Omega \in \mathsf{QD}\left(\frac{c}{w-w_0}\right), \qquad c, w_0 \in \mathbb{C}$$

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$$\exists \ \Omega \in \mathsf{QD}\left(\frac{c}{w-w_0}\right) \ \mathsf{iff} \ |w_0|^2 + 2\mathsf{Re}(c) > 2|c|,$$

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Briefly:

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• For each w_0,c , there is a $t_*>0$ such that $\nexists\Omega\in\mathsf{QD}\left(rac{c}{w-w_0}
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 - $\{\Omega_t\}_{t=0}^{t_*}$ is a Hele-Shaw chain

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 - $\{\Omega_t\}_{t=0}^{t_*}$ is a Hele-Shaw chain
- If $c \not\geq 0$, then $\partial \Omega$ has (3,2)-cusp when $A(\Omega) = t_*$
- Riemann map:

$$\varphi(z) = az \frac{z - z_1}{z - \overline{z_0} - 1}, \qquad (\varphi(z_0) = w_0)$$

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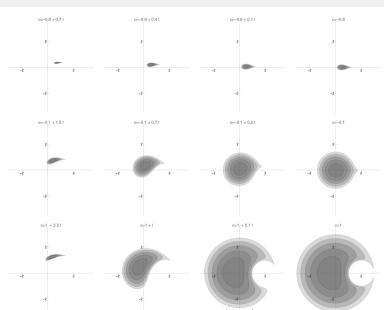
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Antiholomorphic dynamics

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Future work

If Ω is a QD, can consider dynamics of Schwarz reflection, $\{\sigma^{\circ n}\}$, $\sigma = \overline{S}$.

¹³Lee, Lyubich, Makarov & Mukherjee (2018)

Antiholomorphic dynamics

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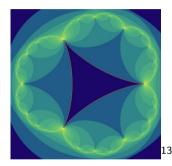
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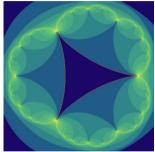
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13

Lee et al. (2018): deltoid S-reflection is the unique conformal mating of \overline{z}^2 and ideal triangle group reflection.

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Antiholomorphic dynamics

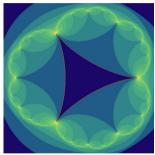
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Lee & Makarov (2016): dynamics of S-reflection → sharp QD connectivity bounds

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